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Entropy for $SU(3)_c$ quark states

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Abstract. We discuss the quantum state structure using the standard model for three colored quarks in the fundamental representations of $SU(3)_c$ making up the singlet ground state of the hadrons. This allows us to calculate a finite von Neumann entropy from the quantum reduced density matrix, which we explicitly evaluate for the quarks in a model for the meson and baryon states.

The well-known heat theorem of Nernst, which is often referred to as the Third Law of Thermodynamics, has the generally accepted interpretation in the theory of gases that the entropy vanishes in the zero temperature limit. Schrödinger [1] had pointed out long ago that when two states contribute to the ground state of a many particle system that a finite constant term could appear in the entropy. In particular, for a system with 2^N states making up the ground state of a system of N particles one should expect a ground state entropy of $N \ln 2$. We can now understand his result in terms of the $SU(2)$ symmetry for the N particles. In this sense we should expect an internal symmetry provided by the quantum structure to yield an entropy following the prescription of von Neumann [2].

The standard model has the color charge carried by the quarks as the fundamental property of the strong nuclear interaction [3]. In clear contrast to the other known charges the color charge cannot be easily isolated and separately measured. In nature it always appears as part of selective states of the $SU(3)_c$, wherein the quarks and antiquarks are placed in the fundamental $\mathbf{3}$ and antifundamental $\mathbf{3}^*$ representations of this group. These two representations together with the adjoint representation of $SU(3)_c$ make up the symmetry structure of quantum chromodynamics (QCD) [4]. The two main categories of strong interacting particles (hadrons) are the mesons, which may be written as a product of the fundamental and the antifundamental representations $\mathbf{3}\otimes\mathbf{3}^*$ and the baryons, which are a product of three fundamental representations $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$. Although the different quarks have other properties like spin, electrical charge and mass, as well as a very special property called *flavor*, we shall not presently go into these aspects here [3].

In this work we shall write the quark and antiquark color states as follows: $|0\rangle$,$|1\rangle$,$|2\rangle$ and $|0^*\rangle$,$|1^*\rangle$,$|2^*\rangle$. We shall use this notation to describe the orthonormal bases of the fundamental and the antifundamental representations of $SU(3)_c$ instead of the more common color names. From these color states we can construct a representation for the color hadronic wavefunctions – in particular for the singlet meson $\Psi_{M,s}$ and baryon $\Psi_{B,s}$ groundstates. We also mention the construction for the eight density matrices for the color octet states of the mesons and baryons. From these color wavefunctions we are able to construct the corresponding density matrices $\rho_{M,s}$ and $\rho_{B,s}$ for the color singlet states [5]. In the following work we shall arrive at the single quark reduced density matrix ρ_q, which is of particular interest in all further calculations. From ρ_q we can directly calculate the quantum entropy in the sense of von Neumann [2, 5]. The results of this calculation show a significant contribution of order one to the entropy of the quarks in the hadronic singlet and octet states. This value is given as a pure number without physical dimensions when we use the usual high energy units with $\hbar$, c and Boltzmann's constant k all set to the value one.

The starting point for the ground state is the evaluation of the density matrix [5] for the singlet quark structure. Here we only consider the color part of the wavefunctions $\Psi_{M,s}$ and $\Psi^*_{M,s}$ coming from the representation $\mathbf{3}\otimes\mathbf{3}^*$ for the color singlet wavefunctions of the mesons,

$$\Psi_{M,s} = \frac{1}{\sqrt{3}}(|00^*\rangle + |11^*\rangle + |22^*\rangle), \tag{1}$$

and keeping the left to right order of the quark and antiquark for its conjugate wavefunction,

$$\Psi^*_{M,s} = \frac{1}{\sqrt{3}}(\langle 00^*| + \langle 11^*| + \langle 22^*|). \tag{2}$$

Similarly we may write a wavefunction for the baryons $\Psi_{B,s}$ coming from the representation $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ for the color singlet state of the baryons,

$$\Psi_{B,s} = \frac{1}{\sqrt{6}}(|012\rangle + |120\rangle + |201\rangle - |021\rangle$$

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$$-|102\rangle - |210\rangle). \quad (3)$$

The conjugate state wavefunction in the order of the tensor product for the baryons is given by

$$\Psi^*_{B,s} = \frac{1}{\sqrt{6}}(\langle 012| + \langle 120| + \langle 201| - \langle 021| - \langle 102| - \langle 210|). \quad (4)$$

We can now write down the density matrices ρ for the hadrons using the direct product of Ψ and Ψ^*. This gives for the color singlet mesons and baryons the density matrices in the following forms:

$$\rho_{M,s} = \Psi_{M,s}\Psi^*_{M,s} \quad (5)$$

and

$$\rho_{B,s} = \Psi_{B,s}\Psi^*_{B,s}. \quad (6)$$

Until now we have only considered the hadronic states as being made out of the quark and antiquark states. The resulting density matrices are for the hadrons *pure* states [5]. However, for the quarks we look at the single quark *reduced* density matrices, which give the statistical state of the individual quark within the hadron. In order to get the reduced density matrices for the mesons, we project out all the antiquark states $\langle i^*|$ and $|j^*\rangle$ by using the orthonormality and the completeness properties. Similarly for the baryons we project onto the other two quark states resulting in two contributions for each color. Thus the meson and the baryon reduced density matrices for the quark states take on the same form:

$$\rho_q = \frac{1}{3}(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|). \quad (7)$$

This is the reduced density matrix for the quarks in the color singlet state. It yields a completely mixed state where each color contribution has the same eigenvalue λ_i equal to the value 1/3. The reduced density matrices can also be calculated for each quark state in the octet representations. A more detailed discussion will appear in a later work.

We can calculate the entropy S of the quantum states using the prescription of von Neumann [2,5], which makes direct use of the density matrix ρ. It is simply written as

$$S = -\mathrm{Tr}(\rho \ln \rho), \quad (8)$$

where the trace "Tr" is taken over the quantum states. When, as is presently the case, the eigenvectors are known for ρ, we may write this form of the entropy in terms of the sum of the eigenvalues λ_i as follows:

$$S = -\sum_i \lambda_i \ln \lambda_i. \quad (9)$$

It is obviously important to have positive eigenvalues. For a zero eigenvalue we use the fact that $x \ln x$ vanishes in the small x limit. Then for the density matrix ρ we may interpret λ_i as the probabilitiy of the state i or p_i. This meaning demands that $0 < p_i \leq 1$. Thus the orthonormality condition for the given states results in the trace condition

$$\mathrm{Tr}\rho = \sum_i p_i = 1. \quad (10)$$

This is a very important condition for the entropy.

We now apply these definitions to the entropy for the quark states. It is clear that the original hadron states are pure colorless states which possess *zero* entropy. For the meson it is immediately obvious since each colored quark state has the opposing colored antiquark state for the resulting colorless singlet state. The sum of all the cycles determine the colorlessness of the baryon singlet state thereby giving no entropy. However, the reduced density matrix for the individual quarks (antiquarks) ρ_q or $\rho_{\bar{q}}$ has a finite entropy. For $SU(3)_c$ all the eigenvalues λ_i from ρ_q have the same value 1/3. Thus we find for all the quarks (antiquarks) in singlet states

$$S_q = \ln\ 3. \quad (11)$$

As a further point we may draw a qualitative comparison of this result for the singlet state with the entropies of the quark octet states. The octet density matrices $\rho_{o,i}$ may be constructed from the eight Gell-Mann matrices $(\lambda)_i$ with $i = 1, 2, \ldots, 8$. The density matrix for each state is constructed by using the properties of $\Psi(\lambda)_i\Psi^*$. From the reduced density matrix, the first seven of these all give the same value ln 2 for the entropy $S_{o,i}$, since all of these states are constructed only from the Pauli matrices. However, the eighth diagonal Gell-Mann matrix yields a larger entropy $S_{o,8}$ from the fact that it involves all three colors although not with equal weights as it was the case for the color singlet state.

Hereupon, we may discuss the entropy in some more detail for the main examples of the colorless hadronic ground states – the mesons and the baryons. As we have discussed above for the density matrix, all the mesons consist of a quark–antiquark pair bound together as a sum of all the three colors. Since each single quark or antiquark state is equally weighted in the reduced density matrix, each state posesses equal probability of 1/3. Thus we easily get the entropy of ln 3. The baryon has the doubly reduced density matrix for each single quark state appearing twice so that with the normalization factor of 1/6 the probability of each colored quark state is again 1/3, which yields the same result for the entropy, ln 3. This value gives the maximal entropy for a completely mixed state. We know that the singlet and the octet states make up a major contribution to all the hadronic states.

It is the colored quark entropy density which physically distinguishes the thermodynamics of the baryons from the mesons. If we use the generally accepted values for the root mean squared charge radius of the hadrons [3], we take for the mesons (pions) $\sqrt{\langle r_M^2 \rangle}$ as 0.66 ± 0.02 fm, while for the baryons (protons) [6] we use for $\sqrt{\langle r_B^2 \rangle}$ the value 0.870 ± 0.008 fm. These values of the charge radii are small when put on the nuclear size scale, where we would generally expect sizes well over 1 fm. These mean charge radii give spherical charge volumes for the mesons ranging from $1.098\,\mathrm{fm}^3$ to $1.317\,\mathrm{fm}^3$ or about $1.20\,\mathrm{fm}^3$ as the average mean volume. Similarly for the baryons we find an average mean volume of $2.76\,\mathrm{fm}^3$. The mean entropy density of the quarks in the singlet ground state of the mesons (pions) is given by

$$s_M = \frac{\ln 3}{1.20\,\mathrm{fm}^3} = 0.912\frac{1}{\mathrm{fm}^3}. \qquad (12)$$

Similarly for the baryons in the singlet ground state we arrive at a mean value for the entropy density

$$s_B = \frac{\ln 3}{2.76\,\mathrm{fm}^3} = 0.398\frac{1}{\mathrm{fm}^3}. \qquad (13)$$

These entropy densities represent the most probable distributions of quarks in the given volume for the charged portions of the hadrons.

As a last remark in this work on the meaning of this type of entropy for the quantum ground state we should note that the effects of this type appear in other systems with internal symmetries. In quantum spin chains [7] the effects of the ground state entanglement show strong correlations in a block of L spins giving entropies proportional to the logarithm of the size L in the various special cases of the quantum Heisenberg model. These results are then related to the entropy in a $(1+1)$-dimensional conformal field theory. Furthermore, one could, perhaps, extend these results to a three state model like the $Z(3)$ symmetric spin models or the extended Potts models [8] to find analogous properties for the entropy in the low temperature limit.

We have calculated the entropy for a single quark in the color singlet ground state of the hadrons. We saw that the singlet state is a completely mixed state with the maximum value of the entropy given by $\ln 3$. In the theory of information it is known that the completely mixed state is that of minimal information, which is consistent with the idea of confinement. If we were to consider a system of N quarks or antiquarks in a gas of hadrons in the same sense that Schrödinger [1] considered a gas of N degenerate two level constituents, we would generally expect an entropy of the form $N\ln 3$ for the uncolored singlet ground state.

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